

# Localization of the Grover walks on spidernets and free Meixner laws

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**Abstract.** A spidernet is a graph obtained by adding large cycles to an almost regular tree and considered as an example having intermediate properties of lattices and trees in the study of discrete-time quantum walks on graphs. We introduce the Grover walk on a spidernet and its one-dimensional reduction. We derive an integral representation of the  $n$ -step transition amplitude in terms of the free Meixner law which appears as the spectral distribution. As an application we determine the class of spidernets which exhibit localization. Our method is based on quantum probabilistic spectral analysis of graphs.

## 1 Introduction

The study of quantum walks, tracing back to [13, 26], has been accelerated from various aspects during the last decade, see e.g., [4, 19, 23, 34] and references cited therein. From a mathematical viewpoint sharp contrast between quantum walks and random walks is of particular importance. For example, the ballistic spreading is observed in a wide class of quantum walks [2, 10, 17, 21, 22, 24, 35], i.e., the speed of a quantum walker's spreading is proportional to the time  $n$  while the typical scale for a random walk is  $\sqrt{n}$ . Moreover, the limit distributions of quantum walks are obtained [10, 17, 21, 22, 24, 31, 35] with a significant contrast with the normal Gaussian law in the case of random walks. In this paper we focus on the phenomenon called *localization*, which is also considered as a typical property of quantum walks, see [8, 17, 25] among others. We introduce the Grover walk on a particular infinite graph called a *spidernet*, consider an isotropic initial state, and determine the class of spidernets which exhibits localization. A spidernet is not only a new example for the localization but also is expected to be a clue to understand localization from graph structure. Our method is based on quantum probabilistic spectral analysis of graphs [15].

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A spidergraph is obtained by adding large cycles to an almost regular tree, see Subsection 3.1 for definition and see Fig. 1 for illustration. It is expected to have intermediate properties between trees and lattices, and its spectral properties have been studied to some extent, see e.g., [16] for the spectral distribution of the adjacency matrix and [33] for estimates of the Cheeger constant and Green kernel in terms of spectra. Then the standard application of the Karlin-McGregor formula (see e.g., [27]) yields an explicit formula for the  $n$ -step transition probability of the isotropic random walk on a spidergraph, where the free Meixner law appears as the spectral distribution. This argument is along a natural extension of the result on random walks on a homogeneous tree due to Kesten [20]. Our attempt in this paper is to establish the quantum counterpart.

In Section 2 we introduce the Grover walk on a general graph after the standard literatures, see e.g., [4, 36]. Then we formulate two concepts of localization, that is, *initial point localization* and *exponential localization*. Several quantum walks are known to exhibit the localization, see e.g., [8, 10, 17, 24, 25, 35]. For relevant discussion see also [29].

In Section 3 we introduce the spidergraph  $S(a, b, c)$  and mention the main results. We first obtain the integral representation of the  $n$ -step transition amplitude for the Grover walk on a spidergraph:

$$\langle \psi_0^+, U^n \psi_0^+ \rangle = \int_{-1}^1 \cos n\theta \mu(d\lambda), \quad n = 0, \pm 1, \pm 2, \dots, \quad (1.1)$$

where  $\lambda = \cos \theta$  and  $\mu$  is the free Meixner law of which the parameters are determined by  $a, b, c$  of the spidergraph under consideration, see Theorem 2 for the precise statement. The free Meixner law is a probability distribution on  $[-1, 1]$  which is the sum of absolutely continuous part and at most two point masses. It is then rather easy to derive from (1.1) the asymptotic behavior of the transition amplitude as  $n \rightarrow \infty$ . In fact, only the effect of the point masses remains in the limit and the asymptotic results follow. In particular, we prove that the initial point localization occurs if and only if  $b > c + \sqrt{c}$ , see Theorem 3 for details.

It is instructive to consider the family of spidergraphs  $S(\kappa, \kappa + 2, \kappa - 1)$ ,  $\kappa \geq 2$ . These are obtained by suitably adding a large cycle to the homogeneous tree with degree  $\kappa$ . We see from Theorem 3 that the initial point localization occurs for  $2 \leq \kappa < 10$  and no initial point localization occurs for  $\kappa \geq 10$  (Corollaries 4 and 5). While, Corollary 6 asserts that no initial point localization occurs on a homogeneous tree either. In the recent work [35] we know that the Grover walk on two-dimensional lattice exhibits the initial point localization. These results suggest the effect of cycles for the localization of the Grover walk.

In Section 4 we introduce the one-dimensional reduction of our Grover walk, called a  $(p, q)$ -quantum walk on  $\mathbb{Z}_+$ . We determine the eigenvalues of the  $(p, q)$ -quantum walk with space cutoff by extending the quantum probabilistic method together with theory of Jacobi matrices.

In Section 5 we obtain the integral expression of the  $n$ -step transition amplitude of the  $(p, q)$ -quantum walk on  $\mathbb{Z}_+$  (Theorem 13) and the asymptotic behavior of the transition amplitude (Theorem 14). With these preparations we prove the main results.

In Appendix we recall the definition of the free Meixner law and derive the associated orthogonal polynomials. The explicit form of the orthogonal polynomials is used to derive the somehow amazing result (Lemma 17) which plays a key role in deriving the exponential localization.

Finally, we mention some relevant works. The so-called CGMV method [7, 8, 14, 25] is also based on the spectral analysis on the unit circle and seems to have close connection with our approach. The technique to get the eigensystem of some class of quantum walks on a finite system including the Grover walk is established in [32]. Our result is an extension to an infinite system, where the orthogonal polynomials with respect to the free Meixner play a key role. Conservation of probability is an interesting question for a quantum walk, see e.g., [10, 17, 24, 30, 35]. The quantum walks studied in [10, 17, 24] are non-conservative and the “missing probability” is found through the weak convergence theorem in such a way that the limit distribution is a convex combination of a point mass at the origin corresponding to localization and the Konno density function [21, 22] coming from ballistic spreading, see [24] for details. It is not yet checked whether our Grover walks are conservative or not. There is a large number of literatures under the name of quantum graphs, see e.g., [12] and references cited therein, which are expected to have a profound relation to quantum walks but not yet very clear.

## 2 Grover Walks on Graphs

Let  $G$  be a graph with vertex set  $V = V(G)$  and edge set  $E = E(G)$ , i.e.,  $V$  is a non-empty (finite or infinite) set and  $E$  is a subset of  $\{\{u, v\}; u, v \in V, u \neq v\}$ . We often write  $u \sim v$  for  $\{u, v\} \in E$ . Throughout the paper a graph is always assumed to be *locally finite*, i.e.,  $\deg(u) = |\{v \in V; v \sim u\}| < \infty$  for all  $u \in V$ , and *connected*, i.e., every pair of vertices are connected by a walk. An ordered pair  $(u, v) \in V \times V$  is called a *half-edge* extending from  $u$  to  $v$  if  $u \sim v$ . Let  $A(G)$  denote the set of half edges of  $G$ .

The state space of our Grover walk will be given by the Hilbert space  $\mathcal{H} = \mathcal{H}(G) = \ell^2(A(G))$  of square-summable functions on  $A(G)$ . The inner product is defined by

$$\langle \phi, \psi \rangle = \sum_{(u,v) \in A(G)} \overline{\phi(u,v)} \psi(u,v), \quad \phi, \psi \in \mathcal{H}.$$

In general, a unit vector in  $\mathcal{H}$  is called a *state*. The canonical orthonormal basis is denoted by  $\{\delta_{(u,v)}; (u,v) \in A(G)\}$ . For  $u \in V$  let  $\mathcal{H}_u$  be the closed subspace spanned by  $\{\delta_{(u,v)}; v \sim u\}$ . Obviously, we have  $\dim \mathcal{H}_u = \deg(u)$  and the orthogonal decomposition:

$$\mathcal{H} = \sum_{u \in V} \oplus \mathcal{H}_u.$$

We next introduce unitary operators on  $\mathcal{H}$ . With each  $u \in V$  we associate a *Grover operator*  $H^{(u)}$  on  $\mathcal{H}_u$  defined by means of the actions on the orthonormal basis  $\{\delta_{(u,v)}; v \sim u\}$ :

$$(H^{(u)})_{vw} \equiv \langle \delta_{(u,v)}, H^{(u)} \delta_{(u,w)} \rangle = \frac{2}{\deg(u)} - \delta_{vw}. \quad (2.2)$$

As is easily verified, the Grover operator  $H^{(u)}$  is a real symmetric, unitary operator on  $\mathcal{H}_u$ . Then the *coin flip operator*  $C$  on  $\mathcal{H}$  is defined by

$$C \delta_{(u,v)} = \sum_{w \sim u} (H^{(u)})_{vw} \delta_{(u,w)}. \quad (2.3)$$

The *shift operator*  $S$  is defined by

$$S\delta_{(u,v)} = \delta_{(v,u)}.$$

Note that  $C^2 = S^2 = I$  (the identity operator). Since both  $C$  and  $S$  are unitary operators on  $\mathcal{H}$ , so is

$$U = SC,$$

which is called the *Grover walk* on the graph  $G$ .

The time evolution of the Grover walk with an initial state  $\Phi_0 \in \mathcal{H} = \ell^2(A(G))$  is given by the sequence of unit vectors:

$$\Phi_n = U^n \Phi_0, \quad n = 0, 1, 2, \dots$$

Since  $U^n$  is unitary, we have

$$1 = \|\Phi_n\|^2 = \sum_{u \in V} \sum_{v \sim u} |\Phi_n(u, v)|^2, \quad n = 0, 1, 2, \dots$$

Therefore, the function

$$u \mapsto \sum_{v \sim u} |\Phi_n(u, v)|^2, \quad u \in V,$$

defines a probability distribution on  $V$ , which is interpreted as the probability of finding a Grover walker at  $u \in V$  at time  $n$ . Following convention we write

$$P(X_n = u) = \sum_{v \sim u} |\Phi_n(u, v)|^2, \quad u \in V. \quad (2.4)$$

It is noted, however, that  $X_n$  is merely defined as a random variable for each  $n$ . It is an interesting question to construct a discrete-time stochastic process  $\{X_n; n = 0, 1, 2, \dots\}$  with state space  $V$  reasonably reflecting probabilistic properties of the Grover walk. The quantity  $\Phi_n(u, v) = \langle \delta_{(u,v)}, U^n \Phi_0 \rangle$  appearing in (2.4), or more generally  $\langle \Phi, U^n \Phi_0 \rangle$  for two states  $\Phi, \Phi_0$  is called a *transition amplitude*. This is a quantum counterpart of transition probability of a Markov chain.

Since the sequence  $\{P(X_n = u); n = 0, 1, 2, \dots\}$  defined in (2.4) is oscillating in general, it is essential to study the time average:

$$\bar{q}^{(\infty)}(u) = \lim_{N \rightarrow \infty} \frac{1}{N} \sum_{n=0}^{N-1} P(X_n = u), \quad u \in V,$$

when the limit exists. For a state  $\Phi \in \mathcal{H} = \ell^2(A(G))$  we denote by  $\text{supp } \Phi$  the set of vertices  $u \in V$  such that  $\Phi(u, v) = \langle \delta_{(u,v)}, \Phi \rangle \neq 0$  for some  $v \sim u$ .

**Definition 1 (Initial point localization)** Let  $o \in V$  be a distinguished vertex and  $\Phi_0 \in \mathcal{H} = \ell^2(A(G))$  a state with  $\text{supp } \Phi_0 = \{o\}$ . We say that the Grover walk on  $G$  with an initial state  $\Phi_0$  exhibits *initial point localization* if  $\bar{q}^{(\infty)}(o) > 0$ .

**Definition 2 (Exponential localization)** Let  $o \in V$  and  $\Phi_0$  be the same as in Definition 1. We say that the Grover walk with an initial state  $\Phi_0$  exhibits *exponential localization* if there exist constant numbers  $C > 0$  and  $0 < r < 1$  such that

$$\bar{q}^{(\infty)}(u) \geq Cr^{\partial(o,u)}, \quad u \in V, \quad (2.5)$$

where  $\partial(o, u)$  stands for the graph distance between  $o$  and  $u$ , i.e., the length of the shortest path connecting them.

**Remark 1** In some literatures, e.g., [1, 18], “exponential localization” is defined when the opposite inequality  $\bar{q}^{(\infty)}(u) \leq Cr^{\partial(o,u)}$  is satisfied instead of (2.5). This concept is more likely referred to as “exponentially bounded” or “exponential decay” and is used for a different purpose.

Note that the concepts of localization in Definitions 1 and 2 depend on the choice of an initial state.

## 3 Main Results

### 3.1 Spidernets

Let  $G = (V, E)$  be a (locally finite and connected) graph with a distinguished vertex  $o \in V$ . We introduce a stratification of  $G$  by

$$V = \bigcup_{j=0}^{\infty} V_j, \quad V_j = \{u \in V; \partial(u, o) = j\}.$$

Then for  $\epsilon \in \{+, -, \circ\}$  we define a function  $\omega_\epsilon$  on  $V$  by

$$\omega_\epsilon(u) = |\{v \in V_{j+\epsilon}; v \sim u\}|, \quad u \in V_j,$$

where we understand  $j + \epsilon = j + 1, j - 1, j$  for  $\epsilon = +, -, \circ$ , respectively. Note that  $\deg(u) = \omega_+(u) + \omega_\circ(u) + \omega_-(u)$  for  $u \in V$ .

A graph is called a *spidernet* if there exist a distinguished vertex  $o \in V$  and integers  $a, b, c$  with

$$a \geq 1, \quad b \geq 2, \quad 1 \leq c \leq b - 1 \quad (3.6)$$

such that

$$\begin{aligned} \omega_+(u) &= \begin{cases} a, & \text{for } u = o, \\ c, & \text{otherwise,} \end{cases} & \omega_-(u) &= \begin{cases} 0, & \text{for } u = o, \\ 1, & \text{otherwise,} \end{cases} \\ \omega_\circ(u) &= \begin{cases} 0, & \text{for } u = o, \\ b - c - 1, & \text{otherwise.} \end{cases} \end{aligned}$$

Such a spidernet is denoted by  $S(a, b, c)$ . It is noted that  $S(a, b, c)$  is not necessarily determined uniquely by the parameters  $a, b, c$ . By definition we have

$$\deg(u) = \begin{cases} a, & \text{for } u = o, \\ b, & \text{otherwise,} \end{cases} \quad (3.7)$$

and

$$|V_0| = 1, \quad |V_j| = ac^{j-1}, \quad j = 1, 2, \dots$$

Hence a spiderweb is an infinite graph.

A spiderweb  $S(a, b, b-1)$  is a tree. In particular,  $S(\kappa, \kappa, \kappa-1)$  with  $\kappa \geq 2$  is the homogeneous tree of degree  $\kappa$ . While, a spiderweb  $S(\kappa, \kappa+2, \kappa-1)$  is obtained by adding a large cycle to each stratum of the homogeneous tree of degree  $\kappa$ . A typical example is shown in Fig. 1; however, note that a spiderweb  $S(\kappa, \kappa+2, \kappa-1)$  is not uniquely determined by the parameter  $\kappa \geq 2$ .

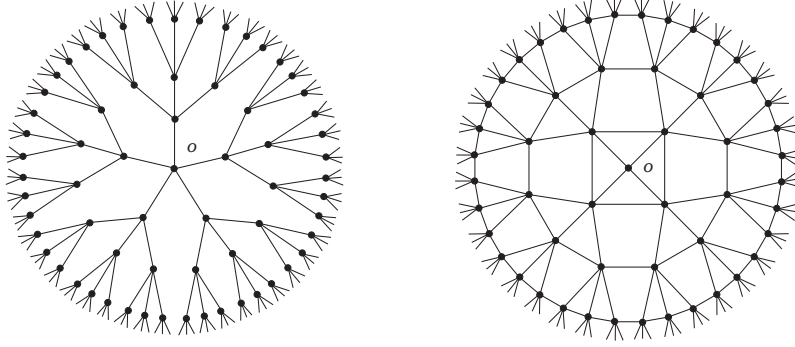


Figure 1:  $S(5, 4, 3)$  and  $S(4, 6, 3)$

### 3.2 Grover walks on spiderwebs

We focus on the Grover walk  $U$  on a spiderweb  $G = S(a, b, c)$ . Recall that the state space is given by  $\mathcal{H} = \ell^2(A(G))$ , of which the canonical orthonormal basis is denoted by  $\{\delta_{(u,v)}; (u,v) \in A(G)\}$ . Define a state  $\psi_0^+ \in \mathcal{H}$  by

$$\psi_0^+ = \frac{1}{\sqrt{a}} \sum_{v \sim o} \delta_{(o,v)}, \quad (3.8)$$

which is taken to be the initial state of our Grover walk. Note that  $\psi_0^+$  is characterized by  $\text{supp } \psi_0^+ = \{o\}$  and being isotropic.

We now list the main results of this paper.

**Theorem 2 (Integral representation of transition amplitude)** *Let  $U$  be the Grover walk on a spiderweb  $S(a, b, c)$  with an initial state  $\psi_0^+$  defined by (3.8). Let  $\mu$  be the free Meixner law with parameters  $q, pq, r$ , where*

$$p = \frac{c}{b}, \quad q = \frac{1}{b}, \quad r = \frac{b-c-1}{b}. \quad (3.9)$$

*Then for all  $n = 0, \pm 1, \pm 2, \dots$  it holds that*

$$\langle \psi_0^+, U^n \psi_0^+ \rangle = \int_{-1}^1 \cos n\theta \mu(d\lambda), \quad \lambda = \cos \theta. \quad (3.10)$$

For the definition of the free Meixner law, see Appendix. It is also noted that (3.10) admits an alternative expression:

$$\langle \psi_0^+, U^n \psi_0^+ \rangle = \int_{-1}^1 T_{|n|}(\lambda) \mu(d\lambda), \quad n = 0, \pm 1, \pm 2, \dots,$$

where  $T_n$  is the *Chebyshev polynomials of the first kind* defined by

$$\cos n\theta = T_n(\cos \theta), \quad n = 0, 1, 2, \dots,$$

see e.g., [9, 15].

From Theorem 2 we will derive some results on initial point localization of our Grover walks. In fact, we determine the class of spidernets  $S(a, b, c)$  which exhibit initial point localization as follows.

**Theorem 3** *Let  $U$  be the Grover walk on a spidernet  $S(a, b, c)$  with an initial state  $\psi_0^+$  defined by (3.8). It then holds that*

$$\langle \psi_0^+, U^n \psi_0^+ \rangle \sim w \cos n\tilde{\theta}, \quad \text{as } n \rightarrow \infty,$$

where

$$w = \max \left\{ \frac{(b-c)^2 - c}{(b-c)(b-c+1)}, 0 \right\}, \quad \cos \tilde{\theta} = -\frac{1}{b-c}, \quad 0 < \tilde{\theta} < \pi.$$

In particular, if  $b > c + \sqrt{c}$ , then the initial point localization occurs:

$$\bar{q}^{(\infty)}(o) = \lim_{N \rightarrow \infty} \frac{1}{N} \sum_{n=0}^{N-1} P(X_n = o) = \frac{w^2}{2} > 0.$$

If  $b \leq c + \sqrt{c}$ , then no localization occurs:

$$\lim_{n \rightarrow \infty} P(X_n = o) = 0 \quad \text{and} \quad \bar{q}^{(\infty)}(o) = 0.$$

It is instructive to consider the family of spidernets  $S(\kappa, \kappa + 2, \kappa - 1)$ ,  $\kappa \geq 2$ . Note that  $S(\kappa, \kappa + 2, \kappa - 1)$  is obtained by adding a large cycle to each stratum of  $S(\kappa, \kappa, \kappa - 1)$ , which is the homogeneous tree of degree  $\kappa$ . Below we list some results obtained immediately from Theorem 3.

**Corollary 4** *Let  $2 \leq \kappa < 10$ . For the Grover walk on a spidernet  $S(\kappa, \kappa + 2, \kappa - 1)$  with initial state  $\psi_0^+$  it holds that*

$$P(X_n = o) = |\langle \psi_0^+, U^n \psi_0^+ \rangle|^2 \sim \left( \frac{10 - \kappa}{12} \right)^2 \cos^2 n\tilde{\theta}, \quad n \rightarrow \infty,$$

where  $\cos \tilde{\theta} = -1/3$ ,  $0 \leq \tilde{\theta} \leq \pi$ . Moreover,

$$\bar{q}^{(\infty)}(o) = \frac{1}{2} \left( \frac{10 - \kappa}{12} \right)^2, \tag{3.11}$$

which means that the Grover walk under consideration exhibits initial point localization. (An example for  $\kappa = 4$  is shown in Fig. 2.)

**Corollary 5** *Let  $\kappa \geq 10$ . For the Grover walk on a spider net  $S(\kappa, \kappa + 2, \kappa - 1)$  with an initial state  $\psi_0^+$  it holds that*

$$\lim_{n \rightarrow \infty} P(X_n = o) = 0, \quad \text{hence} \quad \bar{q}^{(\infty)}(o) = 0.$$

**Corollary 6** *For the Grover walk  $U$  on a spider net  $S(a, b, b - 1)$  with an initial state  $\psi_0^+$  we have*

$$\lim_{n \rightarrow \infty} P(X_n = o) = 0, \quad \text{hence} \quad \bar{q}^{(\infty)}(o) = 0.$$

From Corollaries 4–6 we see that the localization occurs when the “density” of large cycles is high. Further study in this direction is now in progress.

Corollary 6 follows directly from Theorem 2 as a homogeneous tree is a special case of spider nets. While, quantum walks on a tree have been studied from various aspects and the result in Corollary 6 is already known [10]. Note also that localization may occur for the Grover walk on a tree with a non-isotropic initial state.

In relation to Theorem 3 we have the following

**Theorem 7** *Consider a spider net  $S(a, b, c)$  with  $b > c + \sqrt{c}$ . Then for the Grover walk  $U$  with an initial state  $\psi_0^+$  it holds that*

$$\liminf_{N \rightarrow \infty} \frac{1}{N} \sum_{n=0}^{N-1} P(X_n \in V_l) \geq \frac{b}{2c} \left\{ \frac{(b-c)^2 - c}{(b-c)(b-c+1)} \right\}^2 \left\{ \frac{c}{(b-c)^2} \right\}^l, \quad l \geq 1.$$

*If the spider net  $S(a, b, c)$  is rotationally symmetric around  $o$ , we have*

$$\liminf_{N \rightarrow \infty} \frac{1}{N} \sum_{n=0}^{N-1} P(X_n = u) \geq \frac{b}{2a} \left\{ \frac{(b-c)^2 - c}{(b-c)(b-c+1)} \right\}^2 \left\{ \frac{1}{(b-c)^2} \right\}^{\partial(u,o)},$$

*for all  $u \in V$ ,  $u \neq o$ . Namely, the Grover walk under consideration exhibits exponential localization.*

Specializing the parameters in Theorem 7, we obtain the following result with no difficulty.

**Corollary 8** *For  $2 \leq \kappa < 10$  the Grover walk on a spider net  $S(\kappa, \kappa + 2, \kappa - 1)$  with an initial state  $\psi_0^+$  it holds that*

$$\liminf_{N \rightarrow \infty} \frac{1}{N} \sum_{n=0}^{N-1} P(X_n \in V_l) \geq \frac{\kappa + 2}{2(\kappa - 1)} \left( \frac{10 - \kappa}{12} \right)^2 \left( \frac{\kappa - 1}{9} \right)^l, \quad l \geq 1.$$

*If the spider net  $S(\kappa, \kappa + 2, \kappa - 1)$  is rotationally symmetric around  $o$ , we have*

$$\liminf_{N \rightarrow \infty} \frac{1}{N} \sum_{n=0}^{N-1} P(X_n = u) \geq \frac{\kappa + 2}{2\kappa} \left( \frac{10 - \kappa}{12} \right)^2 \left( \frac{1}{9} \right)^{\partial(0,u)}, \quad u \in V, \quad u \neq o.$$

*Namely, the Grover walk under consideration exhibits exponential localization.*



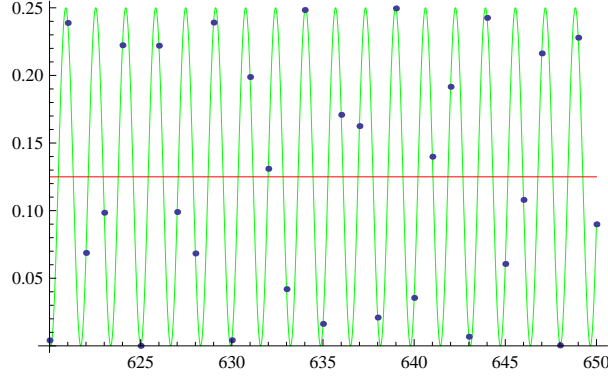


Figure 2: The Grover walk on  $S(4, 6, 3)$  from time  $n = 620$  to  $n = 650$  (see Corollary 4): The dots stand for  $P(X_n = o)$  calculated by numerical simulation. The curve is the graph of  $(1/4) \cos^2(t\tilde{\theta})$  and the horizontal line depicts the time averaged limit probability  $\bar{q}^{(\infty)}(o)$ .

**Remark 9** By changing variable as  $\lambda = \cos \theta$ , the right-hand side of (3.10) becomes an integral over  $[0, \pi]$ . Then using symmetric extension we can write

$$\int_{-1}^1 \cos n\theta \mu(d\lambda) = \int_{-\pi}^{\pi} \cos n\theta \nu(d\theta)$$

with a suitable probability distribution  $\nu$  on  $[-\pi, \pi]$  such that  $\nu(-d\theta) = \nu(d\theta)$ , where no point mass at  $\pm\pi$ . Thus, we have an alternative expression for the transition amplitude:

$$\langle \psi_0^+, U^n \psi_0^+ \rangle = \int_{-\pi}^{\pi} e^{in\theta} \nu(d\theta), \quad n = 0, \pm 1, \pm 2, \dots,$$

which is directly related to the spectral decomposition of the unitary operator  $U$ .

**Remark 10** Let  $\{X_n; n = 0, 1, 2, \dots\}$  be the isotropic random walk on  $S(a, b, c)$  with transition matrix  $T$ . It then follows from the well-established general theory that

$$P(X_n = o | X_0 = o) = \langle \delta_o, T^n \delta_o \rangle = \int_{-1}^1 \lambda^n \mu(d\lambda), \quad n = 0, 1, 2, \dots, \quad (3.12)$$

where  $\mu$  is the same probability distribution as in Theorem 2, see also [27] for relevant discussion along quantum probability. We see that (3.12) makes a good contrast to the transition amplitude (3.10).

## 4 One-dimensional reduction

### 4.1 $(p, q)$ -Quantum walk on $\mathbb{Z}_+$

Let  $U = SC$  be the Grover walk on the spiderweb  $G = S(a, b, c)$ . Define orthonormal vectors in  $\mathcal{H} = \ell^2(A(G))$  by

$$\psi_n^+ = \frac{1}{\sqrt{ac^n}} \sum_{u \in V_n} \sum_{\substack{v \in V_{n+1} \\ v \sim u}} \delta_{(u,v)}, \quad n \geq 0, \quad (4.13)$$

$$\psi_n^\circ = \frac{1}{\sqrt{a(b-c-1)c^{n-1}}} \sum_{u \in V_n} \sum_{\substack{v \in V_n \\ v \sim u}} \delta_{(u,v)}, \quad n \geq 1, \quad (4.14)$$

$$\psi_n^- = \frac{1}{\sqrt{ac^{n-1}}} \sum_{u \in V_n} \sum_{\substack{v \in V_{n-1} \\ v \sim u}} \delta_{(u,v)}, \quad n \geq 1. \quad (4.15)$$

We keep the same notations as in (3.9):

$$p = \frac{c}{b}, \quad q = \frac{1}{b}, \quad r = \frac{b-c-1}{b}, \quad (4.16)$$

verifying that

$$p > 0, \quad q > 0, \quad r = 1 - p - q \geq 0.$$

**Lemma 1** *It holds that*

$$C\psi_n^+ = \begin{cases} \psi_0^+, & n = 0, \\ (2p-1)\psi_n^+ + 2\sqrt{pr}\psi_n^\circ + 2\sqrt{pq}\psi_n^-, & n \geq 1, \end{cases} \quad (4.17)$$

$$C\psi_n^\circ = 2\sqrt{pr}\psi_n^+ + (2r-1)\psi_n^\circ + 2\sqrt{qr}\psi_n^-, \quad n \geq 1, \quad (4.18)$$

$$C\psi_n^- = 2\sqrt{pq}\psi_n^+ + 2\sqrt{qr}\psi_n^\circ + (2q-1)\psi_n^-, \quad n \geq 1. \quad (4.19)$$

*Proof.* By definition we have

$$C\delta_{(x,y)} = \sum_{w \sim x} (H^{(x)})_{yw} \delta_{(x,w)}, \quad (x, y) \in A(G),$$

$$(H^{(x)})_{yw} = \begin{cases} \frac{2}{a} - \delta_{yw}, & x = o, \\ \frac{2}{b} - \delta_{yw}, & \text{otherwise.} \end{cases}$$

We first show (4.17) for  $n = 0$ . Suppose  $(o, y) \in A(G)$ . Then,

$$\begin{aligned} C\delta_{(o,y)} &= \sum_{w \sim o} (H^{(o)})_{yw} \delta_{(o,w)} \\ &= \sum_{w \sim o} \left( \frac{2}{a} - \delta_{yw} \right) \delta_{(o,w)} \\ &= \frac{2}{a} \sum_{w \sim o} \delta_{(o,w)} - \delta_{(o,y)}. \end{aligned}$$

Taking the summation over  $y \sim o$ , we obtain

$$\sum_{y \sim o} C \delta_{(o,y)} = 2 \sum_{w \sim o} \delta_{(o,w)} - \sum_{y \sim o} \delta_{(o,y)} = \sum_{w \sim o} \delta_{(o,w)},$$

from which the desired relation follows by dividing both sides by  $\sqrt{a}$ .

We next prove (4.17) for  $n \geq 1$ . Suppose  $x \in V_n$  with  $n \geq 1$ . Then by definition,

$$\begin{aligned} \sum_{\substack{y \in V_{n+1} \\ y \sim x}} C \delta_{(x,y)} &= \sum_{\substack{y \in V_{n+1} \\ y \sim x}} \sum_{w \sim x} \left( \frac{2}{b} - \delta_{yw} \right) \delta_{(x,w)} \\ &= \frac{2c}{b} \sum_{w \sim x} \delta_{(x,w)} - \sum_{\substack{y \in V_{n+1} \\ y \sim x}} \delta_{(x,y)}, \end{aligned}$$

where  $|\{y \in V_{n+1}; y \sim x\}| = c$  is taken into account. Taking the summation over  $x \in V_n$ , we obtain

$$\begin{aligned} \sum_{x \in V_n} \sum_{\substack{y \in V_{n+1} \\ y \sim x}} C \delta_{(x,y)} &= \frac{2c}{b} \sum_{x \in V_n} \sum_{w \sim x} \delta_{(x,w)} - \sum_{x \in V_n} \sum_{\substack{y \in V_{n+1} \\ y \sim x}} \delta_{(x,y)} \\ &= \frac{2c}{b} \left( \sqrt{ac^n} \psi_n^+ + \sqrt{a(b-c-1)c^{n-1}} \psi_n^\circ \right. \\ &\quad \left. + \sqrt{ac^{n-1}} \psi_n^- \right) - \sqrt{ac^n} \psi_n^+(x) \end{aligned}$$

and then, dividing both sides by  $\sqrt{ac^n}$ , we come to

$$C \psi_n^+ = \left( \frac{2c}{b} - 1 \right) \psi_n^+ + \frac{2\sqrt{c(b-c-1)}}{b} \psi_n^\circ + \frac{2\sqrt{c}}{b} \psi_n^-,$$

which shows (4.17). The rest of the relations is proved in a similar manner. □

**Lemma 2** *It holds that*

$$S \psi_n^+ = \psi_{n+1}^-, \quad n \geq 0, \tag{4.20}$$

$$S \psi_n^\circ = \psi_n^\circ, \quad n \geq 1, \tag{4.21}$$

$$S \psi_n^- = \psi_{n-1}^+, \quad n \geq 1. \tag{4.22}$$

*Proof.* By Straightforward calculation similar to the proof of Lemma 1. □

It is convenient to study the actions of  $C$  and  $S$  described in Lemmas 1 and 2 in a slightly more general context. We consider the Hilbert space  $\mathcal{H}(\mathbb{Z}_+)$  of the form:

$$\mathcal{H}(\mathbb{Z}_+) = \mathbb{C} \psi_0^+ \oplus \sum_{n=1}^{\infty} \oplus (\mathbb{C} \psi_n^+ \oplus \mathbb{C} \psi_n^\circ \oplus \mathbb{C} \psi_n^-),$$

where  $\psi_0^+, \psi_1^+, \psi_1^\circ, \psi_1^-, \dots$  form an orthonormal basis of  $\mathcal{H}(\mathbb{Z}_+)$ . Let  $p, q, r$  be constant numbers satisfying

$$p > 0, \quad q > 0, \quad r = 1 - p - q \geq 0.$$

We then define the coin operator  $C$  and the shift operator  $S$  on  $\mathcal{H}(\mathbb{Z}_+)$  by (4.17)–(4.19) and by (4.20)–(4.22), respectively. It is easily seen that both  $C$  and  $S$  are unitary operators. Hence  $U = SC$  is also a unitary operator on  $\mathcal{H}(\mathbb{Z}_+)$ , which is called the  $(p, q)$ -quantum walk on  $\mathbb{Z}_+$ .

Thus the Grover walk on a spiderweb  $G = S(a, b, c)$  restricted to the closed subspace spanned by  $\{\psi_n^+, n \geq 0\} \cup \{\psi_n^\circ; n \geq 1\} \cup \{\psi_n^-; n \geq 1\}$  is a  $(p, q)$ -quantum walk on  $\mathbb{Z}_+$ , where  $p, q$  are given by (4.16).

We define orthonormal vectors in  $\mathcal{H}(\mathbb{Z}_+)$  by

$$\begin{aligned} \Psi_0 &= \psi_0^+, \\ \Psi_n &= \sqrt{p} \psi_n^+ + \sqrt{r} \psi_n^\circ + \sqrt{q} \psi_n^-, \quad n \geq 1, \end{aligned}$$

and set

$$\Gamma(\mathbb{Z}_+) = \sum_{n=0}^{\infty} \oplus \mathbb{C} \Psi_n.$$

Then  $\Gamma(\mathbb{Z}_+) \subset \mathcal{H}(\mathbb{Z}_+)$  is a closed subspace. Let  $\Pi : \mathcal{H}(\mathbb{Z}_+) \rightarrow \Gamma(\mathbb{Z}_+)$  denote the orthogonal projection.

**Lemma 3** *It holds that*

$$C = C^* = 2\Pi - I.$$

*In particular,  $C$  is the reflection with respect to  $\Gamma(\mathbb{Z}_+)$  and acts on  $\Gamma(\mathbb{Z}_+)$  as the identity.*

*Proof.* Straightforward by definition. □

## 4.2 $(p, q)$ -Quantum walk on a path of finite length

Let  $U$  be a  $(p, q)$ -quantum walk on  $\mathbb{Z}_+$  as in the previous section. We will introduce a  $(p, q)$ -quantum walk on the path of length  $N \geq 2$ , obtained from the  $(p, q)$ -quantum walk on  $\mathbb{Z}_+$  by cutoff.

For  $N \geq 2$  we define a Hilbert space:

$$\mathcal{H}(N) = \mathbb{C} \psi_0^+ \oplus \sum_{n=1}^{N-1} \oplus (\mathbb{C} \psi_n^+ \oplus \mathbb{C} \psi_n^\circ \oplus \mathbb{C} \psi_n^-) \oplus \mathbb{C} \psi_N^-$$

and unitary operators  $C = C_N$  and  $S = S_N$  respectively as in (4.17)–(4.19) and in (4.20)–(4.22), except

$$C \psi_N^- = \psi_N^-. \tag{4.23}$$

Then we obtain a unitary operator  $U = U_N = S_N C_N$  on  $\mathcal{H}(N)$ , which is called the  $(p, q)$ -quantum walk on the path of length  $N$ . Both endpoints play as reflection barriers in analogy of random walks. From now on we omit the suffix  $N$  whenever there is no danger of confusion.

In view of (4.17)–(4.19), (4.23) and (4.20)–(4.22) the explicit actions of  $U$  on  $\psi_j^\epsilon$  are easily written down as follows:

$$U\psi_j^+ = \begin{cases} \psi_1^-, & j = 0, \\ (2p-1)\psi_{j+1}^- + 2\sqrt{pr}\psi_j^\circ + 2\sqrt{pq}\psi_{j-1}^+, & 1 \leq j \leq N-1, \end{cases} \quad (4.24)$$

$$U\psi_j^\circ = 2\sqrt{pr}\psi_{j+1}^- + (2r-1)\psi_j^\circ + 2\sqrt{qr}\psi_{j-1}^+, \quad 1 \leq j \leq N-1, \quad (4.25)$$

$$U\psi_j^- = \begin{cases} 2\sqrt{pq}\psi_{j+1}^- + 2\sqrt{qr}\psi_j^\circ + (2q-1)\psi_{j-1}^+, & 1 \leq j \leq N-1, \\ \psi_{N-1}^+, & j = N, \end{cases} \quad (4.26)$$

The goal of this subsection is to determine the spectra (eigenvalues) of  $U$ . We start with the following result.

**Lemma 4**  $\text{Tr } U = (2r-1)(N-1)$ .

*Proof.* We see from (4.24)–(4.26) that

$$\text{Tr } U = \sum_{\epsilon, n} \langle \psi_n^\epsilon, U\psi_n^\epsilon \rangle = \sum_{j=1}^{N-1} \langle \psi_j^\circ, U\psi_j^\circ \rangle = (2r-1)(N-1)$$

as desired. □

Define orthonormal vectors in  $\mathcal{H}(N)$  by

$$\begin{aligned} \Psi_0 &= \psi_0^+, \\ \Psi_j &= \sqrt{p}\psi_j^+ + \sqrt{r}\psi_j^\circ + \sqrt{q}\psi_j^-, \quad 1 \leq j \leq N-1, \\ \Psi_N &= \psi_N^- \end{aligned}$$

and set

$$\Gamma(N) = \sum_{j=0}^N \oplus \mathbb{C}\Psi_j.$$

Then  $\Gamma(N)$  is a closed subspace of  $\mathcal{H}(N)$  and let  $\Pi = \Pi_N : \mathcal{H}(N) \rightarrow \Gamma(N)$  denote the orthogonal projection. The assertion of Lemma 3 remains true, i.e., it holds that

$$C = C^* = 2\Pi - I.$$

For the  $(p, q)$ -quantum walk  $U = U_N$  we consider  $\Pi U \Pi$  as an operator on  $\Gamma(N)$ , which is denoted by  $T = T_N$ . Thus,

$$T = \Pi U \Pi|_{\Gamma(N)} = \Pi S C \Pi|_{\Gamma(N)} = \Pi S|_{\Gamma(N)}. \quad (4.27)$$

Moreover, by direct calculation we obtain its matrix expression with respect to the orthonormal basis  $\{\Psi_j; 0 \leq j \leq N\}$  as follows:

$$T = T_N = \begin{bmatrix} 0 & \sqrt{q} & & & & & \\ \sqrt{q} & r & \sqrt{pq} & & & & \\ & \sqrt{pq} & r & \sqrt{pq} & & & \\ & & \ddots & \ddots & \ddots & & \\ & & & \sqrt{pq} & r & \sqrt{pq} & \\ & & & & \sqrt{pq} & r & \sqrt{p} \\ & & & & & \sqrt{p} & 0 \end{bmatrix}.$$

For example,

$$\begin{aligned} T\Psi_0 &= \sqrt{q}\Psi_1, \\ T\Psi_1 &= \sqrt{pq}\Psi_2 + r\Psi_1 + \sqrt{q}\Psi_0, \quad \text{etc.} \end{aligned}$$

**Lemma 5** *Every eigenvalue of  $T$  is simple. Moreover,  $\text{Spec}(T) \subset [-1, 1]$  and  $1 \in \text{Spec}(T)$ .*

*Proof.* That every eigenvalue of  $T$  is simple follows from general theory of Jacobi matrices (see e.g., [15, 11]). Since  $T = \Pi S|_{\Gamma(N)}$  by (4.27), the operator norm of  $T$  is bounded by one. Hence every eigenvalue of  $T$  lies in  $[-1, 1]$ . Finally, it is easily verified by expansion that  $\det(T - I) = 0$ . □

**Lemma 6** (1) *If  $r > 0$ , there exists no non-zero  $v \in \Gamma(N)$  such that  $Sv = -v$ .*

(2) *If  $r = 0$ , there exists non-zero  $v \in \Gamma(N)$  such that  $Sv = -v$ . Moreover, such a non-zero vector  $v$  is determined uniquely up to a constant factor.*

*Proof.* Every  $v \in \Gamma(N)$  is in the form:

$$\begin{aligned} v &= \sum_{j=0}^N \gamma_j \Psi_j \\ &= \gamma_0 \psi_0^+ + \sum_{j=1}^{N-1} \gamma_j (\sqrt{p} \psi_j^+ + \sqrt{r} \psi_j^\circ + \sqrt{q} \psi_j^-) + \gamma_N \psi_N^-, \end{aligned}$$

where  $\gamma_0, \dots, \gamma_N$  are constant numbers. Then the equation  $Sv = -v$  is equivalent to the one for these constant numbers, which is obtained by direct calculation:

$$\gamma_1 = -\frac{1}{\sqrt{q}} \gamma_0, \quad \gamma_N = -\sqrt{p} \gamma_{N-1}, \quad (4.28)$$

$$\gamma_j = -\sqrt{\frac{p}{q}} \gamma_{j-1}, \quad 2 \leq j \leq N-1, \quad (4.29)$$

$$\gamma_j \sqrt{r} = -\gamma_j \sqrt{r}, \quad 1 \leq j \leq N-1. \quad (4.30)$$

If  $r > 0$ , it follows from (4.30) that  $\gamma_1 = \dots = \gamma_{N-1} = 0$ . Then in view of (4.28) we also have  $\gamma_0 = \gamma_N = 0$ , which implies  $v = 0$ . If  $r = 0$ , the recurrence relations (4.28) and (4.29) determine the sequence  $\gamma_0, \gamma_1, \dots, \gamma_N$  uniquely by the initial value  $\gamma_0$ . Hence  $\dim\{v \in \Gamma(N); Sv = -v\} = 1$  as desired.

□

**Lemma 7** *If  $Tv = \pm v$  for  $v \in \Gamma(N)$ , then  $Sv = Uv = \pm v$ .*

*Proof.* It is sufficient to consider the case of  $v \neq 0$ . Suppose that  $Tv = \pm v$  for  $v \in \Gamma(N)$ . From  $T = \Pi S|_{\Gamma(N)}$  and  $\Pi v = v$  we obtain  $\Pi(Sv \mp v) = 0$ . Hence  $\langle Sv \mp v, v \rangle = 0$ , which implies that

$$\langle Sv, v \rangle = \pm \langle v, v \rangle.$$

Since  $S$  is unitary, the above relation implies the Schwartz equality and  $Sv = \alpha v$  with some constant  $\alpha \in \mathbb{C}$ . It follows by applying  $\Pi$  that  $\alpha = \pm 1$  and  $Sv = \pm v$ . Finally, since  $U = SC$  by definition and  $C$  acts on  $\Gamma(N)$  as the identity, we have  $Uv = SCv = Sv$ .

□

**Lemma 8** *We have  $-1 \notin \text{Spec}(T)$  for  $r > 0$ , and  $-1 \in \text{Spec}(T)$  for  $r = 0$ .*

*Proof.* Suppose that  $r > 0$  and  $Tv = -v$  for  $v \in \Gamma(N)$ . We see from Lemma 7  $Sv = -v$ , then applying Lemma 6 we come to  $v = 0$ . This means that  $-1$  is not an eigenvalue of  $T$ .

We next suppose that  $r = 0$ . By Lemma 6 there exists a non-zero vector  $v \in \Gamma(N)$  such that  $Sv = -v$ . Then  $Tv = \Pi Sv = -v$ , which means that  $-1$  is an eigenvalue of  $T$ .

□

Thus, the eigenvalues of  $T$  are arranged in such a way that

$$\begin{aligned} \lambda_0 = 1 = \cos \theta_0, \quad \lambda_1 = \cos \theta_1, \quad \lambda_2 = \cos \theta_2, \quad \dots, \quad \lambda_N = \cos \theta_N, \\ 0 = \theta_0 < \theta_1 < \theta_2 < \dots < \theta_N \leq \pi, \end{aligned} \quad (4.31)$$

where  $\theta_N < \pi$  for  $r > 0$  and  $\theta_N = \pi$  for  $r = 0$ . For each eigenvalue  $\lambda_j$  we take a normalized eigenvector  $\Omega_j \in \Gamma(N)$ , i.e.,

$$T\Omega_j = \lambda_j \Omega_j, \quad \|\Omega_j\| = 1.$$

Then we have the orthogonal decomposition of  $\Gamma(N)$  in two ways:

$$\Gamma(N) = \sum_{j=0}^N \oplus \mathbb{C}\Psi_j = \sum_{j=0}^N \oplus \mathbb{C}\Omega_j.$$

We next study the subspace

$$\mathcal{L}(N) = \Gamma(N) + S\Gamma(N), \quad (4.32)$$

which is invariant under the actions of  $S$  and  $U$ . In fact, for  $\phi, \psi \in \Gamma(N)$  we have

$$\begin{aligned} U(\phi + S\psi) &= SC\phi + SC S\psi = S\phi + S(2\Pi - I)S\psi \\ &= S\phi + 2S\Pi S\psi - S^2\psi = S(\phi + 2\Pi S\psi) - \psi, \end{aligned}$$

which shows that  $\mathcal{L}(N) = \Gamma(N) + S\Gamma(N)$  is invariant under  $U$ .

**Lemma 9** (1) If  $r > 0$ , then the vectors  $\Omega_0, \Omega_1, \dots, \Omega_N, S\Omega_1, \dots, S\Omega_N$  are linearly independent. Moreover,

$$\langle \Omega_j, S\Omega_k \rangle = \lambda_k \delta_{jk}, \quad 0 \leq j \leq N, \quad 1 \leq k \leq N. \quad (4.33)$$

(2) If  $r = 0$ , then  $S\Omega_N = -\Omega_N$  and  $\Omega_0, \Omega_1, \dots, \Omega_N, S\Omega_1, \dots, S\Omega_{N-1}$  are linearly independent. Moreover, (4.33) remains valid where  $0 \leq j \leq N$  and  $1 \leq k \leq N-1$ .

*Proof.* (1) Suppose that

$$\alpha_0 \Omega_0 + \sum_{j=1}^N \alpha_j \Omega_j + \sum_{j=1}^N \beta_j S\Omega_j = 0. \quad (4.34)$$

Taking  $\Pi S = T$  in mind, we apply  $\Pi$  to both sides to obtain

$$\alpha_0 \Omega_0 + \sum_{j=1}^N (\alpha_j + \beta_j \lambda_j) \Omega_j = 0. \quad (4.35)$$

Similarly, applying  $\Pi S$  to both sides of (4.34), we obtain

$$\alpha_0 \Omega_0 + \sum_{j=1}^N (\alpha_j \lambda_j + \beta_j) \Omega_j = 0, \quad (4.36)$$

where  $S^2 = I$  and  $S\Omega_0 = \Omega_0$  from Lemma 7 are taken into account. It then follows from (4.35) and (4.36) that

$$\alpha_0 = 0, \quad \alpha_j + \beta_j \lambda_j = \alpha_j \lambda_j + \beta_j = 0, \quad 1 \leq j \leq N.$$

Since  $\lambda_j \neq \pm 1$  for  $1 \leq j \leq N$ , we see that  $\alpha_j = \beta_j = 0$  for all  $j$ . The inner product (4.33) is computed as follows:

$$\langle \Omega_j, S\Omega_k \rangle = \langle \Omega_j, \Pi S\Omega_k \rangle = \langle \Omega_j, T\Omega_k \rangle = \lambda_k \langle \Omega_j, \Omega_k \rangle = \lambda_k \delta_{kj}.$$

(2) is proved similarly by using Lemma 7 and  $\lambda_N = -1$ .

□

**Lemma 10** (1) If  $r > 0$  we have

$$\begin{aligned} U\Omega_0 &= \Omega_0, \\ U\Omega_j &= S\Omega_j, \quad US\Omega_j = -\Omega_j + 2\lambda_j S\Omega_j, \quad 1 \leq j \leq N. \end{aligned}$$

(2) If  $r = 0$ , the above relations hold except  $U\Omega_N = -\Omega_N$ .

*Proof.* Since the proofs are similar we prove only (1). We first observe that

$$U\Omega_j = SC\Omega_j = S(2\Pi - I)\Omega_j = 2S\Pi\Omega_j - S\Omega_j = S\Omega_j, \quad 0 \leq j \leq N.$$



For  $j = 0$  we have  $T\Omega_0 = \Omega_0$  so  $S\Omega_0 = \Omega_0$  by Lemma 7. Hence

$$U\Omega_0 = SC\Omega_0 = \Omega_0.$$

We next calculate  $US\Omega_j$  for  $1 \leq j \leq N$ . Using  $C = 2\Pi - I$  and  $S^2 = I$  we obtain

$$\begin{aligned} US\Omega_j &= SC S\Omega_j = S(2\Pi - I)S\Omega_j \\ &= 2ST\Omega_j - \Omega_j = 2\lambda_j S\Omega_j - \Omega_j. \end{aligned}$$

□

Thus, we obtain the orthogonal decomposition of  $\mathcal{L}(N)$  defined in (4.32):

$$\mathcal{L}(N) = \mathbb{C}\Omega_0 \oplus \sum_{j=1}^N \oplus (\mathbb{C}\Omega_j + \mathbb{C}S\Omega_j), \quad r > 0, \quad (4.37)$$

$$\mathcal{L}(N) = \mathbb{C}\Omega_0 \oplus \sum_{j=1}^{N-1} \oplus (\mathbb{C}\Omega_j + \mathbb{C}S\Omega_j) \oplus \mathbb{C}\Omega_N, \quad r = 0, \quad (4.38)$$

where each factor is invariant under the action of  $U$ .

**Theorem 11** (1) *If  $r > 0$ , the eigenvalues of  $U$  are*

$$1, \quad e^{\pm i\theta_j} \quad (1 \leq j \leq N), \quad -1,$$

*where  $0 < \theta_1 < \theta_2 < \dots < \theta_N < \pi$  are obtained in (4.31). All the eigenvalues except  $-1$  are multiplicity free and the multiplicity of the eigenvalue  $-1$  is  $N - 2$ .*

(2) *If  $r = 0$ , the eigenvalues of  $U$  are*

$$1, \quad e^{\pm i\theta_j} \quad (1 \leq j \leq N - 1), \quad -1,$$

*where  $0 < \theta_1 < \theta_2 < \dots < \theta_{N-1} < \pi$ . All the eigenvalues except  $-1$  are multiplicity free and the multiplicity of the eigenvalue  $-1$  is  $N$ .*

*Proof.* Since the proofs are similar we prove only (1). The orthogonal decomposition (4.37) gives rise to a blockwise diagonalization of  $U$ . It is obvious from Lemma 10 that  $U$  restricted to  $\mathbb{C}\Omega_0$  is the identity operator. Next suppose that  $1 \leq j \leq N$ . We see from Lemma 10 that  $U$  restricted to  $\mathbb{C}\Omega_j + \mathbb{C}S\Omega_j$  admits a matrix representation:

$$\begin{bmatrix} 0 & -1 \\ 1 & 2\lambda_j \end{bmatrix},$$

of which the eigenvalues are

$$\lambda_j \pm i\sqrt{1 - \lambda_j^2} = e^{\pm i\theta_j}.$$

Denoting by  $\mathcal{M}$  the orthogonal complement of  $\mathcal{L}(N)$  in  $\mathcal{H}(N)$ , we have

$$\begin{aligned} \text{Tr}(U) &= 1 + \sum_{j=1}^N 2\lambda_j + \text{Tr}(U|_{\mathcal{M}}) = 1 + 2(\text{Tr}(T) - 1) + \text{Tr}(U|_{\mathcal{M}}) \\ &= 2\text{Tr}(T) - 1 + \text{Tr}(U|_{\mathcal{M}}) = 2r(N - 1) - 1 + \text{Tr}(U|_{\mathcal{M}}). \end{aligned}$$

On the other hand,  $\text{Tr}(U) = (2r - 1)(N - 1)$  by Lemma 4. Hence

$$\text{Tr}(U|_{\mathcal{M}}) = (2r - 1)(N - 1) - (2r(N - 1) - 1) = -(N - 2).$$

Since  $\dim \mathcal{M} = (3N - 1) - (2N + 1) = N - 2$ , we see that  $U|_{\mathcal{M}} = -I$ . Therefore  $\mathcal{M}$  is the eigenspace of  $U$  with eigenvalue  $-1$  so that the multiplicity of the eigenvalue  $-1$  coincides with  $\dim \mathcal{M} = N - 2$ .

□

**Theorem 12** (1) *Let  $r > 0$  and set*

$$\Omega_j^\pm = \frac{1}{\sqrt{2} \sin \theta_j} (\Omega_j - e^{\pm i\theta_j} S \Omega_j), \quad 1 \leq j \leq N.$$

*Then  $\Omega_j^\pm \in (\mathbb{C}\Omega_j + \mathbb{C}S\Omega_j)$ ,  $\|\Omega_j^\pm\| = 1$  and*

$$U\Omega_j^\pm = e^{\pm i\theta_j} \Omega_j^\pm.$$

*In other words,  $\Omega_j^\pm$  is a normalized eigenvector of  $U$  with eigenvalue  $e^{\pm i\theta_j}$ .*

(2) *If  $r = 0$ , the above assertion remains valid for  $1 \leq j \leq N - 1$ .*

*Proof.* That  $\|\Omega_j^\pm\| = 1$  is verified by using Lemma 9. That  $U\Omega_j^\pm = e^{\pm i\theta_j} \Omega_j^\pm$  follows from Lemma 10.

□

## 5 Proofs of main results

### 5.1 Transition amplitudes of the $(p, q)$ -quantum walk on $\mathbb{Z}_+$

For Theorem 2 we need to calculate the transition amplitude:

$$\langle \psi_0^+, U^n \psi_0^+ \rangle = \langle \Psi_0, U^n \Psi_0 \rangle \quad (5.39)$$

for the  $(p, q)$ -quantum walk  $U$  on  $\mathbb{Z}_+$ . A key observation here is that (5.39) is calculated after cutoff. More generally, if  $\phi, \psi \in \mathcal{H}(\mathbb{Z}_+)$  have finite supports,

$$\langle \phi, U^n \psi \rangle = \langle \phi, U_N^n \psi \rangle_{\mathcal{L}(N)} \quad (5.40)$$

holds for all sufficiently large  $N$ . In fact, if  $\text{supp } \phi \subset \{0, 1, \dots, l\}$  and  $\text{supp } \psi \subset \{0, 1, \dots, m\}$ , then (5.40) holds for  $N > \min\{n + l, n + m\}$ . The purpose of this subsection is to derive an integral formula for (5.40).

Now let  $N \geq 2$  be fixed and start with the  $(p, q)$ -quantum walk  $U$  on the path of length  $N \geq 2$ .

**Lemma 11** For  $r > 0$  it holds that

$$\langle \Psi_l, \Omega_j^\pm \rangle = \frac{\mp i e^{\pm i \theta_j}}{\sqrt{2}} \langle \Psi_l, \Omega_j \rangle, \quad 1 \leq j \leq N, \quad (5.41)$$

$$\langle S \Psi_l, \Omega_0 \rangle = \langle \Psi_l, \Omega_0 \rangle, \quad (5.42)$$

$$\langle S \Psi_l, \Omega_j^\pm \rangle = \frac{\mp i}{\sqrt{2}} \langle \Psi_l, \Omega_j \rangle, \quad 1 \leq j \leq N, \quad (5.43)$$

For  $r = 0$  the above relations remain valid for  $1 \leq j \leq N - 1$  and

$$\langle S \Psi_l, \Omega_N \rangle = -\langle \Psi_l, \Omega_N \rangle. \quad (5.44)$$

*Proof.* By definition we have

$$\langle \Psi_l, \Omega_j^\pm \rangle = \frac{1}{\sqrt{2} \sin \theta_j} (\langle \Psi_l, \Omega_j \rangle - e^{\pm i \theta_j} \langle \Psi_l, S \Omega_j \rangle). \quad (5.45)$$

Since  $\Pi \Psi_l = \Psi_l$  we have

$$\langle \Psi_l, S \Omega_j \rangle = \langle \Psi_l, \Pi S \Omega_j \rangle = \langle \Psi_l, T \Omega_j \rangle = \lambda_j \langle \Psi_l, \Omega_j \rangle.$$

Then (5.45) becomes

$$\langle \Psi_l, \Omega_j^\pm \rangle = \frac{1}{\sqrt{2} \sin \theta_j} (1 - e^{\pm i \theta_j} \lambda_j) \langle \Psi_l, \Omega_j \rangle. \quad (5.46)$$

We see easily from  $\cos \theta_j = \lambda_j$  that

$$1 - e^{\pm i \theta_j} \lambda_j = \mp i (\sin \theta_j) e^{\pm i \theta_j}.$$

Inserting the above relation into (5.46), we obtain (5.41).

We next show (5.43). In view of the definition of  $\Omega_j^\pm$  and using  $S^2 = I$  we have

$$\begin{aligned} \langle S \Psi_l, \Omega_j^\pm \rangle &= \frac{1}{\sqrt{2} \sin \theta_j} (\langle S \Psi_l, \Omega_j \rangle - e^{\pm i \theta_j} \langle S \Psi_l, S \Omega_j \rangle) \\ &= \frac{1}{\sqrt{2} \sin \theta_j} (\langle \Psi_l, S \Omega_j \rangle - e^{\pm i \theta_j} \langle \Psi_l, \Omega_j \rangle). \end{aligned}$$

Then, applying a similar consideration as above, we obtain (5.43) with no difficulty.

Finally, since  $U \Omega_0 = S \Omega_0 = \Omega_0$ , we have

$$\langle S \Psi_l, \Omega_0 \rangle = \langle \Psi_l, S \Omega_0 \rangle = \langle \Psi_l, \Omega_0 \rangle,$$

which shows (5.42). For (5.44) we need only to note that  $S \Omega_N = -\Omega_N$ .

□

**Lemma 12** For  $0 \leq l, m \leq N$  and  $n = 0, \pm 1, \pm 2, \dots$  it holds that

$$\langle \Psi_l, U^n \Psi_m \rangle = \sum_{j=0}^N (\cos n\theta_j) \langle \Psi_l, \Omega_j \rangle \langle \Omega_j, \Psi_m \rangle, \quad (5.47)$$

$$\langle S \Psi_l, U^n \Psi_m \rangle = \langle \Psi_l, U^{n-1} \Psi_m \rangle, \quad (5.48)$$

$$\langle \Psi_l, U^n S \Psi_m \rangle = \langle \Psi_l, U^{n+1} \Psi_m \rangle, \quad (5.49)$$

$$\langle S \Psi_l, U^n S \Psi_m \rangle = \langle \Psi_l, U^n \Psi_m \rangle. \quad (5.50)$$

*Proof.* Because the proofs are similar, we prove the assertions under  $r > 0$ . Since  $\Psi_l$  and  $U^n \Psi_m$  are vectors in  $\mathcal{L}(N)$ , the left-hand side of (5.47) is expanded in terms of the orthonormal basis  $\Omega_0, \Omega_1^\pm, \dots, \Omega_N^\pm$  as follows:

$$\begin{aligned} \langle \Psi_l, U^n \Psi_m \rangle &= \langle \Psi_l, \Omega_0 \rangle \langle \Omega_0, U^n \Psi_m \rangle + \sum_{j=1}^N \langle \Psi_l, \Omega_j^+ \rangle \langle \Omega_j^+, U^n \Psi_m \rangle \\ &\quad + \sum_{j=1}^N \langle \Psi_l, \Omega_j^- \rangle \langle \Omega_j^-, U^n \Psi_m \rangle \end{aligned} \quad (5.51)$$

The first term becomes

$$\langle \Psi_l, \Omega_0 \rangle \langle \Omega_0, U^n \Psi_m \rangle = \langle \Psi_l, \Omega_0 \rangle \langle U^{-n} \Omega_0, \Psi_m \rangle = \langle \Psi_l, \Omega_0 \rangle \langle \Omega_0, \Psi_m \rangle. \quad (5.52)$$

For the second term of (5.51) we see that

$$\begin{aligned} \sum_{j=1}^N \langle \Psi_l, \Omega_j^+ \rangle \langle \Omega_j^+, U^n \Psi_m \rangle &= \sum_{j=1}^N \langle \Psi_l, \Omega_j^+ \rangle \langle U^{-n} \Omega_j^+, \Psi_m \rangle \\ &= \sum_{j=1}^N \langle \Psi_l, \Omega_j^+ \rangle \langle e^{-in\theta_j} \Omega_j^+, \Psi_m \rangle \\ &= \sum_{j=1}^N e^{in\theta_j} \langle \Psi_l, \Omega_j^+ \rangle \langle \Omega_j^+, \Psi_m \rangle. \end{aligned}$$

Then we apply Lemma 11 to have

$$\begin{aligned} \sum_{j=1}^N \langle \Psi_l, \Omega_j^+ \rangle \langle \Omega_j^+, U^n \Psi_m \rangle &= \sum_{j=1}^N e^{in\theta_j} \frac{-ie^{i\theta_j}}{\sqrt{2}} \langle \Psi_l, \Omega_j \rangle \frac{ie^{-i\theta_j}}{\sqrt{2}} \langle \Omega_j, \Psi_m \rangle \\ &= \sum_{j=1}^N \frac{e^{in\theta_j}}{2} \langle \Psi_l, \Omega_j \rangle \langle \Omega_j, \Psi_m \rangle. \end{aligned} \quad (5.53)$$

Applying a similar argument to the third term of (5.51), we obtain

$$\sum_{j=1}^N \langle \Psi_l, \Omega_j^- \rangle \langle \Omega_j^-, U^n \Psi_m \rangle = \sum_{j=1}^N \frac{e^{-in\theta_j}}{2} \langle \Psi_l, \Omega_j \rangle \langle \Omega_j, \Psi_m \rangle. \quad (5.54)$$

Summing up (5.52)–(5.54), we see that (5.51) becomes

$$\begin{aligned}\langle \Psi_l, U^n \Psi_m \rangle &= \langle \Psi_l, \Omega_0 \rangle \langle \Omega_0, \Psi_m \rangle + \sum_{j=1}^N \frac{e^{in\theta_j} + e^{-in\theta_j}}{2} \langle \Psi_l, \Omega_j \rangle \langle \Omega_j, \Psi_m \rangle \\ &= \sum_{j=0}^N (\cos n\theta_j) \langle \Psi_l, \Omega_j \rangle \langle \Omega_j, \Psi_m \rangle,\end{aligned}$$

where  $\theta_0 = 0$  is taken into account. Thus, (5.47) is proved.

Noting that  $U = SC$  and  $C$  acts on  $\Gamma(N)$  as the identity, we see that

$$\begin{aligned}\langle S\Psi_l, U^n \Psi_m \rangle &= \langle S\Psi_l, SCU^{n-1} \Psi_m \rangle \\ &= \langle \Psi_l, CU^{n-1} \Psi_m \rangle \\ &= \langle C\Psi_l, U^{n-1} \Psi_m \rangle \\ &= \langle \Psi_l, U^{n-1} \Psi_m \rangle,\end{aligned}$$

which proves (5.48). We next observe that

$$\langle \Psi_l, U^n S\Psi_m \rangle = \langle \Psi_l, U^n SC\Psi_m \rangle = \langle \Psi_l, U^{n+1} \Psi_m \rangle,$$

which proves (5.49). Finally, we see that

$$\langle S\Psi_l, U^n S\Psi_m \rangle = \langle SC\Psi_l, U^n SC\Psi_m \rangle = \langle U\Psi_l, U^{n+1} \Psi_m \rangle = \langle \Psi_l, U^n \Psi_m \rangle,$$

which shows (5.50). □

Let  $\{P_n; n = 0, 1, 2, \dots\}$  be the orthogonal polynomials with respect to the free Meixner law with parameters  $q, pq, r$ , i.e., the polynomials defined by the Jacobi parameters

$$\omega_1 = q, \quad \omega_2 = \omega_3 = \dots = pq; \quad \alpha_1 = 0, \quad \alpha_2 = \alpha_3 = \dots = r,$$

see also Appendix. We set

$$\begin{aligned}p_0(x) &= P_0(x) = 1, \\ p_j(x) &= \frac{P_j(x)}{\sqrt{\omega_1 \dots \omega_j}} = \frac{P_j(x)}{\sqrt{q(pq)^{j-1}}}, \quad j = 1, 2, \dots, N.\end{aligned}\tag{5.55}$$

It is shown that  $\{p_j; 0 \leq j \leq N\}$  satisfies the recurrence relations determined by  $T_N$ . We define

$$\mu_N = \sum_{j=0}^N \rho(j) \delta_{\lambda_j}, \quad \rho(j) = \rho_N(j) = \left( \sum_{n=0}^N p_n(\lambda_j)^2 \right)^{-1}.$$

The following results are known by general theory of Jacobi matrices and orthogonal polynomials. [11, 15]

**Lemma 13**  $\mu_N$  is a probability distribution uniquely determined by the Jacobi matrix  $T_N$ . Moreover,  $\{p_j; j = 0, 1, 2, \dots, N\}$  is the orthogonal polynomials with respect to  $\mu_N$ , normalized so as to have norm one, i.e.,

$$\int_{-1}^1 p_j(x) p_k(x) \mu_N(dx) = \delta_{jk}.$$

**Lemma 14** For  $j = 0, 1, \dots, N$  let  $\Omega_j$  be the normalized eigenvector of  $T_N$  with eigenvalue  $\lambda_j$  such that  $\langle \Omega_j, \Psi_0 \rangle > 0$ . Then,

$$\Omega_j = \sqrt{\rho_N(j)} \sum_{n=0}^N p_n(\lambda_j) \Psi_n,$$

or equivalently,

$$\langle \Omega_j, \Psi_n \rangle = \sqrt{\rho_N(j)} p_n(\lambda_j).$$

The next result is a key for removing the cutoff.

**Lemma 15** The sequence of probability distributions  $\mu_N$  converges weakly to the free Meixner law with parameters  $q, pq, r$ . In particular, for any continuous function  $f$  on  $[-1, 1]$  we have

$$\lim_{N \rightarrow \infty} \int_{-1}^1 f(x) \mu_N(dx) = \int_{-1}^1 f(x) \mu(dx).$$

*Proof.* We first note that

$$\lim_{N \rightarrow \infty} \int_{-\infty}^{+\infty} x^m \mu_N(dx) = \int_{-1}^1 x^m \mu(dx), \quad m = 0, 1, 2, \dots \quad (5.56)$$

In fact, the  $m$ -th moment of  $\mu_N$  is a polynomial in the first  $m$  terms of the Jacobi coefficients of  $\mu_N$ , which are identical with the first  $m$  terms of the Jacobi coefficients of the free Meixner law  $\mu$  if  $m < N$ . Since the free Meixner law has a compact support, it is uniquely determined by the moment sequence. Therefore, it follows by general theory that (5.56) implies the weak convergence of  $\mu_N$  to  $\mu$ . □

**Theorem 13 (Integral representation of transition amplitude)** Let  $U$  be the  $(p, q)$ -quantum walk on  $\mathbb{Z}_+$  and  $\mu$  the free Meixner law with parameters  $q, pq, r$ . For any  $l, m \in \mathbb{Z}_+$  and  $n = 0, \pm 1, \pm 2, \dots$  it holds that

$$\langle \Psi_l, U^n \Psi_m \rangle = \int_{-1}^1 (\cos n\theta) p_l(\lambda) p_m(\lambda) \mu(d\lambda), \quad \cos \theta = \lambda. \quad (5.57)$$

Moreover,

$$\begin{aligned} \langle S \Psi_l, U^n \Psi_m \rangle &= \int_{-1}^1 (\cos(n-1)\theta) p_l(\lambda) p_m(\lambda) \mu(d\lambda), \\ \langle \Psi_l, U^n S \Psi_m \rangle &= \int_{-1}^1 (\cos(n+1)\theta) p_l(\lambda) p_m(\lambda) \mu(d\lambda), \\ \langle S \Psi_l, U^n S \Psi_m \rangle &= \int_{-1}^1 (\cos n\theta) p_l(\lambda) p_m(\lambda) \mu(d\lambda). \end{aligned}$$

*Proof.* Since  $\langle \Psi_l, U^n \Psi_m \rangle$  coincides with the similar expression for  $(p, q)$ -quantum walk on the path of length  $N > \min\{l + n, m + n\}$ . We take such a sufficiently large  $N$ . By Lemmas 12 and 14 we have

$$\begin{aligned} \langle \Psi_l, U^n \Psi_m \rangle &= \sum_{j=0}^N (\cos n\theta_j) \langle \Psi_l, \Omega_j \rangle \langle \Omega_j, \Psi_m \rangle \\ &= \sum_{j=0}^N (\cos n\theta_j) p_l(\lambda_j) p_m(\lambda_j) \rho(j) \\ &= \int_{-1}^1 (\cos n\theta) p_l(\lambda) p_m(\lambda) \mu_N(d\lambda), \end{aligned} \quad (5.58)$$

which holds for all sufficiently large  $N$ . Then, taking Lemma 15 into account, we come to

$$\begin{aligned} \langle \Psi_l, U^n \Psi_m \rangle &= \lim_{N \rightarrow \infty} \int_{-1}^1 (\cos n\theta) p_l(\lambda) p_m(\lambda) \mu_N(d\lambda) \\ &= \int_{-1}^1 (\cos n\theta) p_l(\lambda) p_m(\lambda) \mu(d\lambda). \end{aligned}$$

This completes the proof of (5.57). The rest is proved by combination of Lemma 12 and (5.57). □

**Theorem 14** *Let  $U$  be the  $(p, q)$ -quantum walk on  $\mathbb{Z}_+$  with parameters satisfying*

$$p + q + r = 1, \quad p \geq q > 0, \quad r \geq 0. \quad (5.59)$$

*Then it holds that*

$$\langle \Psi_l, U^n \Psi_0 \rangle \sim w p_l(\xi) \cos n\tilde{\theta}, \quad \text{as } n \rightarrow \infty,$$

*where*

$$w = \max \left\{ \frac{(1-p)^2 - pq}{(1-p)(1-p+q)}, 0 \right\}, \quad \xi = -\frac{q}{1-p} = \cos \tilde{\theta}, \quad 0 < \tilde{\theta} < \pi.$$

*Therefore,*

$$\lim_{N \rightarrow \infty} \frac{1}{N} \sum_{n=0}^{N-1} |\langle \Psi_0, U^n \Psi_0 \rangle|^2 = \frac{w^2}{2}.$$

*In particular,  $U$  exhibits the initial point localization if and only if  $w > 0$ , i.e.,  $(1-p)^2 - pq > 0$ .*

*Proof.* By (5.57) we have

$$\langle \Psi_l, U^n \Psi_0 \rangle = \int_{-1}^1 (\cos n\theta) p_l(\lambda) \mu(d\lambda), \quad \cos \theta = \lambda. \quad (5.60)$$

Under the assumption (5.59) the free Meixner law with parameters  $q, pq, r$  is of the form:

$$\mu(dx) = \rho(x)dx + w\delta_\xi,$$

where  $\rho$  is a continuous function on  $[r - 2\sqrt{pq}, r + 2\sqrt{pq}] \subset [-1, 1]$ , an explicit form is deferred in Appendix, and

$$\xi = -\frac{q}{1-p}, \quad w = \max \left\{ \frac{(1-p)^2 - pq}{(1-p)(1-p+q)}, 0 \right\}.$$

Since  $\rho$  is an integrable function, the Riemann–Lebesgue lemma implies that

$$\lim_{n \rightarrow \infty} \int_{-1}^1 (\cos n\theta) p_l(\lambda) \rho(\lambda) d\lambda = 0.$$

Hence in (5.60) only contribution by the point mass remains in the limit, i.e.,

$$\langle \Psi_l, U^n \Psi_0 \rangle \sim w p_l(\xi) \cos n\tilde{\theta}, \quad \text{as } n \rightarrow \infty,$$

as desired. The rest is straightforward. □

## 5.2 Proof of Theorem 2

Let  $U$  be the Grover walk on a spidernet  $G = S(a, b, c)$  and consider the initial state  $\psi_0^+$  defined by (3.8). Let

$$\Gamma(\mathbb{Z}_+) \subset \mathcal{H}(\mathbb{Z}_+) \subset \mathcal{H}(G)$$

be the subspaces defined in Subsection 4.1. Then  $\mathcal{H}(\mathbb{Z}_+)$  is invariant under  $U$  and  $U|_{\mathcal{H}(\mathbb{Z}_+)}$  is the  $(p, q)$ -quantum walk on  $\mathbb{Z}_+$ , where

$$p = \frac{c}{b}, \quad q = \frac{1}{b}, \quad r = \frac{b-c-1}{b}. \quad (5.61)$$

Since the initial state  $\psi_0^+ = \Psi_0$  belongs to  $\mathcal{H}(\mathbb{Z}_+)$ ,  $\langle \psi_0^+, U^n \psi_0^+ \rangle$  is obtained from the  $(p, q)$ -quantum walk on  $\mathbb{Z}_+$ . In fact, by Theorem 13 we have

$$\langle \psi_0^+, U^n \psi_0^+ \rangle = \int_{-1}^1 (\cos n\theta) \mu(d\lambda), \quad \lambda = \cos \theta, \quad (5.62)$$

where  $\mu$  be the free Meixner law with parameters  $q, pq, r$ . This completes the proof of Theorem 2.

## 5.3 Proof of Theorem 3

For a spidernet  $G = S(a, b, c)$  the parameters  $p, q, r$  defined by (5.61) satisfies the condition in Theorem 14. So it holds that

$$\langle \Psi_0, U^n \Psi_0 \rangle \sim w \cos n\tilde{\theta}, \quad \text{as } n \rightarrow \infty,$$

where

$$w = \max \left\{ \frac{(1-p)^2 - pq}{(1-p)(1-p+q)}, 0 \right\}, \quad (5.63)$$

$$\xi = -\frac{q}{1-p} = \cos \tilde{\theta}, \quad 0 < \tilde{\theta} < \pi. \quad (5.64)$$



For the first half of Theorem 3 it is sufficient to apply the following obvious relations:

$$\frac{(1-p)^2 - pq}{(1-p)(1-p+q)} = \frac{(b-c)^2 - c}{(b-c)(b-c+1)}, \quad -\frac{q}{1-p} = -\frac{1}{b-c}.$$

For the second half we need only to note that  $(b-c)^2 - c > 0$  is equivalent to  $b > c + \sqrt{c}$  under the assumption (3.6) posed at the beginning.

## 5.4 Proofs of Corollaries 4–6

These follow immediately from Theorem 3. We need only to check the parameters. For a spider net  $S(\kappa, \kappa + 2, \kappa - 1)$  we have

$$\xi = \cos \tilde{\theta} = -\frac{1}{b-c} = -\frac{1}{3},$$

$$w = \max \left\{ \frac{(b-c)^2 - c}{(b-c)(b-c+1)}, 0 \right\} = \max \left\{ \frac{10 - \kappa}{12}, 0 \right\}.$$

While, for a spider net  $S(a, b, b - 1)$  we have

$$(b-c)^2 - c = -(b-2) \leq 0, \quad \kappa \geq 2,$$

which implies  $w = 0$ .

## 5.5 Proof of Theorem 8

In a similar manner as in the proof of Theorem 3 we see that

$$\langle \Psi_l, U^n \Psi_0 \rangle \sim w p_l(\xi) \cos n \tilde{\theta}, \quad \text{as } n \rightarrow \infty, \quad (5.65)$$

where  $w, \xi, \tilde{\theta}$  are given by (5.63) and (5.64). The value  $p_l(\xi)$  is known explicitly from Lemma (17) below:

$$p_l(\xi) = \frac{1}{\sqrt{p}} \left( -\frac{\sqrt{pq}}{1-p} \right)^l = \sqrt{\frac{b}{c}} \left( -\frac{\sqrt{c}}{b-c} \right)^l, \quad l = 1, 2, \dots$$

Then the time averaged limit probability is given by

$$\begin{aligned} \lim_{N \rightarrow \infty} \frac{1}{N} \sum_{n=0}^{N-1} |\langle \Psi_l, U^n \Psi_0 \rangle|^2 &= \frac{w^2}{2} p_l(\xi)^2 \\ &= \frac{1}{2} \left\{ \frac{(b-c)^2 - c}{(b-c)(b-c+1)} \right\}^2 \times \frac{b}{c} \left\{ \frac{c}{(b-c)^2} \right\}^l. \end{aligned} \quad (5.66)$$

We here use the following rather obvious result.

**Lemma 16** *Let  $U$  be the Grover walk on a spider net  $S(a, b, c)$  with an initial state  $\psi_0^+$ . Then we have*

$$P(X_n \in V_l) \geq |\langle \Psi_l, U^n \Psi_0 \rangle|^2, \quad l = 1, 2, \dots \quad (5.67)$$

*Proof.* We first note the obvious inequality:

$$\begin{aligned} P(X_n \in V_l) &= \sum_{u \in V_l} \sum_{v \sim u} |\langle \delta_{(u,v)}, U^n \psi_0^+ \rangle|^2 \\ &\geq \left| \left\langle \frac{1}{\sqrt{b|V_l|}} \sum_{u \in V_l} \sum_{v \sim u} \delta_{(u,v)}, U^n \Psi_0 \right\rangle \right|^2. \end{aligned} \quad (5.68)$$

On the other hand, from the definitions (4.13)–(4.15) we see that

$$\sum_{u \in V_l} \sum_{v \sim u} \delta_{(u,v)} = \sqrt{ac^l} \psi_l^+ + \sqrt{a(b-c-1)c^{l-1}} \psi_l^\circ + \sqrt{ac^{l-1}} \psi_l^-.$$

Then, noting that  $|V_l| = ac^{l-1}$  for  $l \geq 1$ , we obtain

$$\frac{1}{\sqrt{b|V_l|}} \sum_{u \in V_l} \sum_{v \sim u} \delta_{(u,v)} = \sqrt{p} \psi_l^+ + \sqrt{r} \psi_l^\circ + \sqrt{q} \psi_l^- = \Psi_l.$$

Inserting the above relation into (5.68), we obtain (5.67). □

Applying Lemma 16 to (5.66), we obtain

$$\liminf_{N \rightarrow \infty} \frac{1}{N} \sum_{n=0}^{N-1} P(X_n \in V_l) \geq \frac{b}{2c} \left\{ \frac{(b-c)^2 - c}{(b-c)(b-c+1)} \right\}^2 \left\{ \frac{c}{(b-c)^2} \right\}^l, \quad (5.69)$$

which proves the first half of Theorem 7. If the spider net  $S(a, b, c)$  is rotationally symmetric around  $o$ , we have

$$P(X_n = u) = \frac{1}{|V_l|} P(X_n \in V_l), \quad \partial(o, u) = l.$$

Then the second half of Theorem 7 follows by dividing (5.69) by  $|V_l| = ab^{l-1}$ .

Finally, we calculate the value of  $p_l(x)$  at  $x = \xi$ . The result is somehow amazing and plays a key role in showing the exponential localization.

**Lemma 17** *Let  $p, q, r$  be constant numbers satisfying*

$$p > 0, \quad q > 0, \quad r = 1 - p - q \geq 0, \quad (1-p)^2 - pq > 0.$$

*Let  $\{p_n\}$  be the orthogonal polynomials associated with the free Meixner law with parameters  $q, pq, r$ , normalized to have norm one as before, see (5.55). Then we have*

$$p_n \left( -\frac{q}{1-p} \right) = \frac{1}{\sqrt{p}} \left( -\frac{\sqrt{pq}}{1-p} \right)^n, \quad n = 1, 2, \dots$$

*Proof.* We see from Theorem 2 that the orthogonal polynomials  $\{P_n\}$  associated with the free Meixner law with parameters  $q, pq, r$  verify

$$P_n(x) = \frac{(xR_+(x) - 2q)R_+(x)^{n-1} - (xR_-(x) - 2q)R_-(x)^{n-1}}{2^{n-1}(R_+(x) - R_-(x))}, \quad n \geq 1, \quad (5.70)$$

where

$$R_{\pm}(x) = x - r \pm \sqrt{(x - r)^2 - 4pq}, \quad (x - r)^2 - 4pq > 0.$$

We need to compute the value of  $P_n(x)$  at  $\xi = -q/(1 - p)$ . Noting first that

$$(\xi - r)^2 - 4pq = \left(-\frac{q}{1 - p} - r\right)^2 - 4pq = \left\{\frac{(1 - p)^2 - pq}{1 - p}\right\}^2,$$

we obtain

$$R_+(\xi) = -\frac{2pq}{1 - p}, \quad R_-(\xi) = -2(1 - p),$$

and hence

$$\begin{aligned} \xi R_+(\xi) - 2q &= \frac{2q(pq - (1 - p)^2)}{(1 - p)^2}, \quad \xi R_-(\xi) - 2q = 0, \\ \xi(R_+(\xi) - R_-(\xi)) &= \xi R_+(\xi) - 2q. \end{aligned}$$

Then putting  $x = \xi$  in (5.70) we have

$$\begin{aligned} P_n(\xi) &= \frac{(\xi R_+(\xi) - 2q)R_+(\xi)^{n-1}}{2^{n-1}(R_+(\xi) - R_-(\xi))} \\ &= \frac{\xi}{2^{n-1}} R_+(\xi)^{n-1} \\ &= \frac{1}{p} \left(-\frac{pq}{1 - p}\right)^n, \quad n = 1, 2, \dots \end{aligned}$$

Finally, in view of (5.55) we have

$$p_n(x) = \frac{P_n(x)}{\sqrt{q(pq)^{n-1}}} = \frac{1}{\sqrt{p}} \left(-\frac{\sqrt{pq}}{1 - p}\right)^n, \quad n = 1, 2, \dots$$

This completes the proof. □

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## Appendix A: Free Meixner laws

The free Meixner law with parameters  $p > 0$ ,  $q \geq 0$ ,  $a \in \mathbb{R}$  is a probability distribution  $\mu$  on  $\mathbb{R}$  uniquely determined by

$$\int_{-\infty}^{+\infty} \frac{\mu(dx)}{z-x} = \frac{1}{z} - \frac{p}{z-z-a} - \frac{q}{z-a} - \frac{q}{z-a} - \dots,$$

where the continued fraction in the right-hand side converges in  $\mathbb{C} \setminus \mathbb{R}$ . In other words,  $\mu$  is uniquely determined by the so-called Jacobi coefficients:

$$\omega_1 = p, \omega_2 = \omega_3 = \dots = q; \quad \alpha_1 = 0, \alpha_2 = \alpha_3 = \dots = a.$$

The free Meixner law with parameters  $p = q = 1$ ,  $a = 0$  is nothing else the (normalized) *Wigner semicircle law* and the one with parameters  $p > 0$ ,  $q \geq 0$ ,  $a = 0$  the *Kesten distribution* [20] with parameters  $p, q$ . The free Meixner laws have been studied mostly in the context of free probability and quantum probability [5, 6, 15, 16, 28].

In general, with Jacobi parameters  $\{\omega_n\}$ ,  $\{\alpha_n\}$  we associate a sequence of polynomials  $\{P_n\}$  by

$$\begin{aligned} P_0(x) &= 1, \\ P_1(x) &= x - \alpha_1, \\ P_n(x) &= P_{n+1}(x) + \alpha_{n+1}P_n(x) + \omega_n P_{n-1}(x), \quad n \geq 1. \end{aligned}$$

It is known that  $\{P_n(x)\}$  is the orthogonal polynomials with respect to  $\mu$ . We will derive an explicit expression of the orthogonal polynomials with respect to the free Meixner law.

Let  $\{U_n\}$  be the Chebyshev polynomials of the second kind, i.e., defined by

$$U_n(\cos \theta) = \frac{\sin(n+1)\theta}{\sin \theta}, \quad n = 0, 1, 2, \dots$$

Set

$$\tilde{U}_n(x) = U_n\left(\frac{x}{2}\right).$$

It is well known that  $\{\tilde{U}_n\}$  form the orthogonal polynomials with respect to the normalized Wigner semicircle law (the free Meixner law with parameters  $p = q = 1$ ,  $a = 0$ ) and are specified uniquely by the recurrence relations:

$$\begin{aligned} \tilde{U}_0(x) &= 1, \\ \tilde{U}_1(x) &= x, \\ x\tilde{U}_n(x) &= \tilde{U}_{n+1}(x) + \tilde{U}_{n-1}(x), \quad n \geq 1. \end{aligned}$$

**Theorem 1** *Let  $p > 0$ ,  $q > 0$  and  $a \in \mathbb{R}$ . The orthogonal polynomial with respect to the free Meixner law with parameters  $p, q, a$  is given by*

$$\begin{aligned} P_0(x) &= 1, \\ P_1(x) &= x, \\ P_n(x) &= q^{n/2} \tilde{U}_n\left(\frac{x-a}{\sqrt{q}}\right) + aq^{(n-1)/2} \tilde{U}_{n-1}\left(\frac{x-a}{\sqrt{q}}\right) \\ &\quad + (q-p)q^{(n-2)/2} \tilde{U}_{n-2}\left(\frac{x-a}{\sqrt{q}}\right), \quad n \geq 2, \end{aligned}$$

*Proof.* Set

$$\begin{aligned} V_0(x) &= 1, \\ V_1(x) &= x, \\ V_n(x) &= q^{n/2} \left\{ \tilde{U}_n\left(\frac{x}{\sqrt{q}}\right) + \left(1 - \frac{p}{q}\right) \tilde{U}_{n-2}\left(\frac{x}{\sqrt{q}}\right) \right\}, \quad n \geq 2. \end{aligned}$$

Then  $V_n(x) = x^n + \dots$  and it holds that

$$\begin{aligned} xV_1(x) &= V_2(x) + pV_0(x), \\ xV_n(x) &= V_{n+1}(x) + qV_{n-1}(x), \quad n \geq 2. \end{aligned}$$

In other words,  $\{V_n\}$  is the orthogonal polynomials with respect to the Kesten distribution with parameters  $p, q$ . Then, it is straightforward to verify that the polynomials  $\{P_n\}$  defined by

$$\begin{aligned} P_0(x) &= 1, \\ P_n(x) &= V_n(x - a) + aq^{(n-1)/2} \tilde{U}_{n-1}\left(\frac{x - a}{\sqrt{q}}\right), \quad n \geq 1, \end{aligned}$$

satisfy

$$\begin{aligned} P_0(x) &= 1, \\ P_1(x) &= x, \\ xP_1(x) &= P_2(x) + aP_1(x) + pP_0(x), \\ xP_n(x) &= P_{n+1}(x) + aP_n(x) + qP_{n-1}(x), \quad n \geq 2. \end{aligned}$$

This means that  $\{P_n\}$  is the orthogonal polynomials with respect to the free Meixner law with parameters  $p, q, a$ .

□

By direct application of the famous expression of the Chebyshev polynomials of the second kind:

$$U_n(x) = \frac{(x + \sqrt{x^2 - 1})^{n+1} - (x - \sqrt{x^2 - 1})^{n+1}}{2\sqrt{x^2 - 1}}, \quad n = 0, 1, 2, \dots,$$

which is valid for  $|x| > 1$ , we obtain a variant of Theorem 1 as follows.

**Theorem 2** *Let  $p > 0, q > 0$  and  $a \in \mathbb{R}$ . The orthogonal polynomial with respect to the free Meixner law with parameters  $p, q, a$  is given by*

$$\begin{aligned} P_0(x) &= 1, \\ P_n(x) &= \frac{(xR_+(x) - 2p)R_+(x)^{n-1} - (xR_-(x) - 2p)R_-(x)^{n-1}}{2^{n-1}(R_+(x) - R_-(x))}, \quad n \geq 1, \end{aligned}$$

where

$$R_{\pm}(x) = x - a \pm \sqrt{(x - a)^2 - 4q}, \quad (x - a)^2 - 4q > 0.$$

Finally, we mention briefly the explicit form of the free Meixner law. For  $p > 0$ ,  $q \geq 0$ ,  $a \in \mathbb{R}$  we set

$$\rho(x) = \frac{p}{2\pi} \frac{\sqrt{4q - (x - a)^2}}{(q - p)x^2 + pax + p^2}, \quad |x - a| \leq 2\sqrt{q}.$$

The free Meixner law is the sum of  $\rho(x)dx$  and at most two atoms:

$$\mu(dx) = \rho(x)dx + w_1\delta_{\xi_1} + w_2\delta_{\xi_2},$$

where  $w_1 \geq 0$ ,  $w_2 \geq 0$  and  $\xi_1 \neq \xi_2$ . For the explicit form, see e.g., [15, 28].